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THE LINEAR CONTINUUM IN TERMS OF POINT AND LIMIT.[*]

BY ROBERT L. MOORE.

Introduction.

F. Riesz[†] has proposed a set of four postulates for a continuum in terms of point and limit in the non-sequential[‡] sense. To these he adds certain postulates involving the third undefined notion "*verkettung.*" His postulates hold true for continua of any number of dimensions and do not form a categorical set. In the present paper I propose a categorical set of eight postulates for the linear continuum[§] in terms of *point* and *limit.* The first three of these postulates (or axioms) are substantially equivalent to the first three of Riesz' system.

The treatment of the linear continuum contained in the present paper connects with Lennes' characterization of a continuous arc AB as a closed, connected set of points containing A and B but containing no connected proper subset that contains A and B.

§ 1. Axioms and Definitions.

AXIOM 1. *If the point set M contains the point set N then every limit point of N is also a limit point of M.*

AXIOM 2. *If M and N are two point sets with no point in common then every limit point of $M + N$*[‖] *is either a limit point of M or a limit point of N.*

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[*] A part of this paper, under a different title, was presented to The American Mathematical Society, April 26, 1914.

[†] F. Riesz, Stetigkeitsbegriff und Mengenlehre, Atti del IV Congresso Internazionale dei Matematici, Roma, 1908, vol. 2, pp. 18–24.

[‡] In his thesis, Frechet postulates limit in the sequential sense. Cf. M. Frechet, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 6. See also E. R. Hedrick, On properties of a domain for which any derived set is closed, Transactions of the American Mathematical Society, vol. 12 (1911), pp. 285–294; and T. H. Hildebrandt, A contribution to the foundations of Frechet's Calculfonctionnel, American Journal of Mathematics, vol. 34 (1912), p. 237.

[§] Postulates for the linear continuum in terms of point and order have been given by O. Veblen, Definition in terms of order alone in the linear continuum and in well ordered sets, Transactions of the American Mathematical Society, vol. 6 (1905), pp. 165–171, and by E. V. Huntington, A set of postulates for a one-dimensional continuum and for the theory of groups, pages 17–41 of the same volume. See also R. E. Root, Limits in terms of point and order, Transactions, vol. 15 (1914), pp. 51–71.

[‖] In the language of point set theory, if M and N are two point sets then $M + N$ denotes the set of all points that belong to either M or N. If M contains N and at least one additional point then $M - N$ denotes the set of all points that belong to M and not to N.

Axiom 3. *No point is a limit point of a single point.*

Definition 1. The point set K is said to be *connected* if, however it be divided into two mutually exclusive* sets, one of those sets contains a limit point of the other one.

Axiom 4. *If P is a point and S is the set of all points, then $S - P$ is the sum of two mutually exclusive connected sets neither of which contains a limit point of the other.*

Definition 2. If P is a point and S is the set of all points, and $S - P = S_P' + S_P''$, where S_P' and S_P'' are mutually exclusive connected point sets such that neither of them contains a limit point of the other one, then S_P' and S_P'' are called rays. If B is a point of the ray S_P', then the ray S_P' is denoted by PB. The ray S_P' is said to be complementary to the ray S_P''.

It follows from Axiom 1 that if $P \neq B$ then there is not more than one ray PB.

Axiom 5. *There do not exist three mutually exclusive rays.*

Axiom 6. *The set of all points is a connected set.*

Axiom 7.† *There exists a countable set of points such that every point either belongs to this set or is a limit point of it.*

Axiom 8. *There exist at least two points.*

In the following AB denotes the ray AB. S denotes the set of all points.

§ 2. Consequences of Axioms 1, 2, 3, 4 and 6.

Theorem 1. *If $B \neq C$, then B is a limit point of the ray BC.*

Proof. By Axiom 4 there exist two points, C and C', such that $S - B = BC + BC'$. Since $S = (B + BC') + BC$, therefore, by Axiom 6, either $S + BC'$ contains a point P which is a limit point of BC or BC contains a point P' which is a limit point of $B + BC'$. But in the latter case, by Axiom 2, either P' is a limit point of B, which contradicts Axiom 3, or P' is a limit point of BC' which is contrary to hypothesis and Definition 2. Hence $B + BC'$ contains a point P which is a limit point of BC. But BC' can contain no limit point of BC. Hence, by Axiom 2, B is a limit point of BC.

Theorem 2. *If $A \neq B$ and A is not on BC, then AB contains BC.*

Proof. If AB does not contain BC then since A is not a point of BC, BC must contain at least one point in common with AB', the complement of AB. Let K be the set of all such common points. There are two cases to be considered.

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* Two or more point sets are said to be mutually exclusive if no two of them have a point in common.

† Compare Axiom 7 with Frechet's definition of a classe séparable, loc. cit., p. 23. Cf. also G. Cantor, Zur Begründung der transfiniten Mengenlehre, I Mathematische Annalen, vol. 46 (1895), p. 510. Veblen's Postulate of Uniformity is *in some respects* weaker than Axiom 7. Cf. O. Veblen, loc. cit., p. 166.

Case I. Suppose that every point of BC is a point of AB'. Then, by Theorem 1, and Axiom 1, B is a limit point of AB' which is contrary to Definition 2.

Case II. Suppose BC contains at least one point that does not belong to AB'. Let K' denote the set of all such points. Then since BC is connected, either K contains a limit point of K' or K' contains a limit point of K. But K is a subset of AB' and, since by hypothesis A is not on BC and therefore is not in K' and furthermore no point of AB' is in K', therefore K' is a subset of AB. Thus again in case II one of the rays AB and AB' would contain a limit point of the other one. But this is contrary to Definition 2.

THEOREM 3. *If $B \neq A$, then every point P lies either on BA or on AB.*

Proof. If P is not on AB, then either P is A or AB and AP are two complementary rays. Hence either P is A or B is not on AP. Therefore, by Theorem 2, BA contains P.

DEFINITION 3. Three points A, B, C are said to be in the order ABC if and only if they are mutually distinct and A is not on BC. The abbreviation "ABC," used as a sentence, signifies that A, B and C are in the order ABC.

THEOREM 4. *If ABC, then CBA.*

Proof. If C were on BA, then BC would be identical with BA and A would be on BC, which is contrary to hypothesis.

THEOREM 5. *If ABC, then not BCA.*

Proof. If B were not on CA, then, by Theorem 2, BC would contain A, which is contrary to hypothesis.

THEOREM 6. *If ABC and BCD, then ACD.*

Proof. Since A is not on BC and B is not on CD, therefore, by Theorem 2, AB contains BC and BC contains CD. Hence AB contains CD. Therefore, if A were on CD, AB would contain A, which is contrary to Definition 2.

THEOREM 7. *If ABC and ACD, then ABD.*

Proof. By hypothesis and Theorem 2, AB contains BC and AC contains CD. Hence AB is identical with AC and AC contains D. Therefore AB contains D. Suppose now that ABD were not true. Then, by Theorem 4, Theorem 5 and Definition 3, D would be on BA. But, by hypothesis, ABC. Hence, by Theorems 2 and 4, CB contains BA and therefore CA contains BA. Hence if D were on BA, D would be on CA. Hence A would be on CD which is contrary to the hypothesis that ACD. Hence D is not on BA. Hence DBA and therefore ABD.

THEOREM 8. *If ABC and BCD, then ABD.*

Proof. From ABC and BCD it follows, by Theorem 6, that ACD. From ACD and ABC it follows, by Theorem 7, that ABD.

THEOREM 9. *If ABC and ACD, then BCD.*

Proof. By hypothesis and Theorem 7, ABD. If BCD were not true, then B would lie on CD. Hence CB would be identical with CD. But ACD. Therefore A is not on CD. Hence A would not be on CB. But ABC. Therefore CBA. Hence, by Theorem 2, A is on CB. Thus the supposition that B is on CD leads to a contradiction.

DEFINITION 4. The abbreviation $ABCD$ signifies that ABC, ABD, ACD and BCD.

THEOREM 10. *If ABC and ACD, or if ABC and BCD, then ABCD.*

THEOREM 11. *If A and B are distinct points, there exists a point C such that ABC.*

Proof. By Axiom 4 there exists at least one point C different from B and not on the ray BA. Hence CBA and therefore ABC, by Theorem 4.

§ 3. Consequences of Axioms 1–6.

THEOREM 12. *If ABC and ABD, and* $C \neq D$, *either BCD or BDC.*

Proof. By hypothesis, Theorem 4 and Definition 3, BA contains neither C nor D. By Theorem 2, BA is in CB and in DB. Suppose now that neither BCD nor BDC. Then B is on CD and on DC and therefore CB is CD and DB is DC. Hence BA is on CD and on DC. By Theorem 11 there exist points C' and D' such that CDC' and DCD'. By Theorem 2, DC contains CD'. Hence if CD' and DC' had a point in common then DC and DC' would have a point in common, which contradicts CDC'. But BA lies in DC and in CD. Hence it can have no point in common with either DC' or CD'. Thus no two of the three rays BA, CD' and DC' have a point in common. But this is contrary to Axiom 5. Hence the supposition that neither BCD nor BDC leads to a contradiction.

THEOREM 13. *If A and C are two distinct points, there exists a point B such that ABC.*

Proof. By Theorem 11 there exists a point D such that ACD. By Theorem 4, DCA. By Theorem 1, A is a limit point of AC. But since A is not C and A is not on CD, therefore A is not a limit point of CD. Hence, by Axioms 1, 2 and 3, AC must contain at least one point B which is different from C and does not belong to CD. Hence BCD and therefore DCB. From DCB and DCA it follows by Theorem 12 that either CAB or CBA. But B is on AC. Hence it is not true that BAC. Therefore CAB is false. Hence CBA and therefore ABC.

THEOREM 14. *If A, B and C are three distinct points, then either ABC, BCA or CAB.*

Proof. Suppose that ABC is not true. Then CBA is not true and therefore C is on BA. By Theorems 11 and 4 there exists a point A' such that $A'BA$. Hence A' is not on BA. But since C is on BA, therefore BA is BC. Therefore A' is not on BC. Hence $A'BC$. Hence, by Theorem 12, either BCA or BAC. Thus it is proved that if ABC is not true, then either BCA or BAC is true.

It is now easy to see that if Axioms 1–6 hold true for a space S then all of Hilbert's linear Axioms of Group II[*] hold true in S with respect to the betweenness defined in Definition 2. In particular it may be shown that any four distinct points may be so lettered A, B, C, D that $ABCD$. If Axioms 7 and 8 are added, my set of axioms becomes[†] categorical with respect to point and betweenness *as defined* in Definition 2. From this fact it is not[‡] *apriori* evident that the set of Axioms 1–8 is *absolutely* categorical (i. e. categorical with respect to point and *limit*, the undefined symbols in terms of which these Axioms are stated). It can however be proved that a point P is a limit point of a point set M if, and only if, every segment that contains P contains at least one point of M different from P. Thus the statement that P is a limit point of M is (in the presence of Axioms 1–8 and Definition 2) equivalent to a statement in terms of point and betweenness. It follows that our set of Axioms is categorical with respect to *point* and *limit* point of a point set.

§ 4. Concerning Axiom 5.

LEMMA 1. *It follows from Axioms 1, 2, 3, 4, 6 and Theorems 12 and 13 that if* $A \neq C$ *and the two rays* AB *and* CD *have no point in common, then* $BACD$.

Proof. By hypothesis, Theorem I, and Axioms 1 and 3 B is not on CD and D is not on AB. Hence BCD and DAB. By Theorem 14[§] either DCA, CAD or ADC.

Suppose that ADC were true. Then CDA. But DAB. Hence CDB, by Theorem 8. Thus ADC is impossible.

Suppose that CAD. By Theorem 13 there exists a point X such that CXA. From CXA and CAD it follows, by Theorem 7, that CXD. From CAD and BCD it follows by Theorems 4 and 9 that ACB. But AXC. Hence, by Theorem 7, AXB. But from CXD and AXB it follows that AB and CD have the point X in common. This is contrary to hypothesis. Thus CAD is false.

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[*] D. Hilbert, The Foundations of Geometry, Translation by E. J. Townsend, Chicago, 1902, pp. 5–7.

[†] Cf. G. Cantor, loc. cit.

[‡] Cf. § 7.

[§] It was shown above that Theorem 14 is a consequence of Axioms 1, 2, 3, 4, 6 and Theorem 12.

It follows that DCA is true. Similarly BAC is true. Hence $BACD$.

LEMMA 2. *Axiom 5 is a Consequence of Axioms 1, 2, 3, 4, 6 and Theorems 12 and 13.*

Proof. Suppose that AB, CD and EF are three mutually exclusive rays. It is clear that A, C and E must be mutually distinct. Hence, by Lemma 1, $BACD$, $BAEF$, $DCEF$ and $DCAB$. From $DCEF$ it follows that $F \neq C$. By Theorem 12, from $BACD$ and $BAEF$ it follows that ACF or AFC and from $DCEF$ and $DCAB$ it follows that CFA or CAF. But, by Theorems 4 and 5, ACF and CAF cannot both be true. Hence, by Theorem 4, AFC. From CAB and AFC it follows, by Theorems 4 and 7, that CFB. From CFB and CEF it follows that EFB and therefore that EF contains B which is contrary to hypothesis. Thus Axiom 5 is proved on the basis of Axioms 1, 2, 3, 4, 6 and Theorems 12 and 13.

THEOREM 15. *In the presence of Axioms 1, 2, 3, 4 and 6, Axiom 5 is equivalent to Theorems 12 and 13 combined.*

§ 5. Questions of Independence.

According to E. H. Moore a set of postulates T is said to be *completely* independent if

(1) For each subset of T there exists a system in which all the postulates of that subset are satisfied and the remaining postulates of T are not satisfied.

(2) There exists a system in which no postulate of the set T is satisfied.

Since the denial of Axiom 8 implies that S contains not more than one point, it would seem very improbable *apriori* that the set 1–8 should be completely independent. If 8 be omitted however the remaining set, G, of Axioms 1–7 is completely independent except that there exist no examples of type E_{567} or E_{2567}. Here the notation E_{567} is used to denote an example of a system in which 5, 6, and 7 are false and the remaining axioms 1, 2, 3 and 4 are true. In general, the letter E with subscripts denotes an example of a system in which those axioms are false whose numbers occur as subscripts of E and the remaining axioms of the set G are true.

The following notation will be used.

K_1 denotes a segment without end points.

K_2 denotes a segment with end points.

K_3 denotes the set of all the interior points of two circles which have no interior point in common.

K_4 denotes the set of all points, except O, on and interior to a circle R with center at O.

K_5 denotes the set of all interior points of two colinear segments which have no point in common.

K_6 denotes a circle C together with three rays DD', EE' and FF' where D, E and F are distinct points of C, and D', E' and F' are three points such that ODD', OEE' and OFF' where O is the center of C.

K_7 is the same as K_6 except that in K_7 all points X such that $OD'X$ are omitted.

K_8 denotes a segment t together with two distinct points neither of which is a point or an end point of t.

K_9 denotes a segment t together with three distinct points no one of which is a point or an end point of t.

For the sake of convenience the following independence examples will be separated into groups.

In each example of group I there are in S three distinct points, A, B and C such that BAC in the ordinary sense and such that: (1) A is a *limit* point of a point set M if, and only if, M contains B and C, or M contains A, or A is a limit point of M in the usual sense.

(2) If $P \neq A$, then P is a *limit* point of M only if it is a limit point of M in the usual sense.

GROUP I.

E_{23}. S is K_1, E_{2345}. S is K_6, E_{2346}. S is K_5,
E_{234}. S is K_2, E_{236}. S is K_3, E_{23456}. S is K_8.

In each example of group II, a point P is a *limit* point of a point set M if, and only if,

(1) P belongs to M or is a limit point of M in the ordinary sense.

(2) M contains at least two points.

GROUP II.

E_2. S is K_1, E_{245}. S is K_6, E_{2456}. S is K_8,
E_{24}. S is K_2, E_{246}. S is K_5, E_{26}. S is K_3.

In each example of group III, P is a limit point of M if, and only if, P is a limit point of M in the usual sense or P belongs to M.

GROUP III.

E_3. S is K_1, E_{36}. S is K_3, E_{346}. S is K_9,
E_{34}. S is K_2, E_{345}. S is K_6, E_{3456}. S is K_8.

In each example of group IV, a point P is a *limit* point of a point set M if, and only if, P is a limit point of M in the usual sense.

GROUP IV.

E_4. S is K_2, E_{45}. S is K_6, E_{456}. S is K_8.
E_6. S is K_3, E_{46}. S is K_5,

GROUP V.

E_{235}. S is K_4. There are three points A, B and C within R such that $OBAC$ in the ordinary sense and such that

(1) A is a *limit* point of a point set M if, and only if, M contains B and C, or M contains A, or A is a limit point in the usual sense of a subset of M which lies on the ray OA.

(2) If $P \neq A$ then P is a *limit* point of M if, and only if, P is a limit point, in the usual sense, of a subset of M that lies on the ray OP, or of a subset of M that lies on R.

E_{2356}. S is composed of the interior, K, of a circle together with two distinct points A and B that lie without that circle. A is a *limit* point of M if, and only if, M contains B. If P is a point in K, then P is a *limit* point of the point set M if, and only if, M contains two or more points of K. B is a *limit* point of no point set.

E_{25}. S is K_4. P is a *limit* point of M if, and only if, (1) P belongs to M or is a limit point in the usual sense of a subset of M that lies entirely on R or entirely on OP, (2) M contains at least two points.

E_{35}. S is K_4. P is a *limit* point of M if, and only if, P belongs to M or is a limit point in the usual sense of a subset of M that lies entirely on R or entirely on the radius OP.

E_5. S is K_4. P is a limit point of M if, and only if, P is a limit point in the usual sense of a subset of M that lies entirely on R or entirely on the radius OP.

E_{356}. S is the same as in E_{2356}. P is a *limit* point of M if, and only if, either, (1) P belongs to K and is a limit point of M in the usual sense or, (2) $P = A$ and M contains B.

GROUP VI.

E_{56}. S is composed of three points no one of which is a *limit* point of any point or set of points.

E_{256}. S is composed of three points A, B and C. P is a *limit* point of M if, and only if, $P = A$ and M is $A + B$ or $A + B + C$.

Let E denote any one of the preceding examples. Let E′ denote an example which is the same as E except that in E′ no point is a *limit* point of S. Then every axiom of the set G which is satisfied in E is also satisfied in E′ and conversely, except that Axiom 1 is not satisfied in E′.

Let $\overline{E}$ denote any one of the preceding examples (examples of type E′ being included) with the exception of the examples E_{56}, E_{256}, E_{56}' and E_{256}'. In $\overline{E}$ either S is an ordinary continuum of one or two dimensions, or it contains such a continuum. If a linear continuum occurs in $\overline{E}$ replace it by a system of the sort described in Veblen's independence example for his uniformity Postulate E.* If a two dimensional continuum exists in $\overline{\mathrm{E}}$ replace it by a certain two-dimensional set† which may be easily constructed with the help of the above-mentioned linear set of Veblen's. Let $\overline{\mathrm{E}}'$ denote an example which is the same as $\overline{\mathrm{E}}$ except for the above mentioned substitution and certain modifications which naturally accompany that substitution. Then every postulate of the set G which holds true in $\overline{\mathrm{E}}$ holds true also in $\overline{\mathrm{E}}'$ and conversely, except that 7 is not true in $\overline{\mathrm{E}}'$.

We have yet to consider the existence or non-existence of E_{1567}, E_{12567}, E_{567} and E_{2567}. The first two of these exist.

E_{1567}. S is composed of three mutually exclusive subsets S_1, S_2, S_3. A *point* is a system of four real numbers $(x_1, y_1; x_2, y_2)$ such that $0 \leqq y_k \leqq 1$ and such that either $0 < x_k < 1$, $2 < x_k < 3$ or $4 < x_k < 5$ $(k = 1, 2)$. In the first case $(x_1, y_1; x_2, y_2)$ belongs to S_1, in the second case to S_2 and in the third case to S_3. The point $(x_1', y_1'; x_2', y_2')$ is a *limit* point of the point set M if, and only if, one of the following two conditions is fulfilled.

Condition I. Corresponding to any preassigned positive number ϵ there exists, in the set M, a point $(x_1'', y_1''; x_2'', y_2'')$, distinct from $(x_1', y_1'; x_2', y_2')$, such that (1) $|x_k' - x_k''| < \epsilon$, $|y_k' - y_k''| < \epsilon$ $(k = 1, 2)$ (2) $|x_k' - x_k''| = 0$ in case y_k' is distinct from 0 and from 1 $(k = 1, 2)$.

Condition II. The point $(x_1', y_1'; x_2', y_2')$ belongs to the set S_i, and M is a non-denumerable proper subset of S_{i+1} (mod. 3) $(i = 1, 2, 3)$.

E_{12567}. This example is the same as E_{1567} except that in E_{12567} every point P in S_1 is a limit point of every pair of points in S_1 which are distinct from each other and from P.

THEOREM 16. *If Axioms 5 and 6 are false and Axioms 1, 3 and 4 are true, then S must consist of just three points.*

Proof. Since Axiom 6 is false, S must consist of two subsets S_1 and S_2 neither of which contains a limit point of the other. Since Axiom 5 is false, there exist three mutually exclusive rays r_1, r_2, r_3. Hence one of the sets S_1 and S_2 contains at least two points. If S_1 contains more than one point and P is a point of S_1 then it easily follows from Axioms 4 and 1 that P divides S into the two rays $S_1 - P$ and S_2 and therefore S_2 is connected. Similarly if S_2 contains more than one point then S_1 is connected. Let P_1, P_2, P_3 denote the origins of the rays r_1, r_2, r_3 respectively. At least two of these points (say P_1 and P_2) must be distinct from each other. There are two cases to consider.

Case I. Suppose P_1 and P_2 are in S_1. Then, by Axioms 4 and 1,

$$r_1 \text{ is either } S_1 - P_1 \text{ or } S_2, \qquad r_2 \text{ is either } S_1 - P_2 \text{ or } S_2.$$

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* O. Veblen, loc. cit., p. 169.

† The set S_1, described below in connection with E_{1567}, may be used for this purpose.

But r_1, r_2 and r_3 are mutually exclusive. It clearly follows that $S_1 = P_1 + P_2$. But, by Axiom 3, $P_1 + P_2$ is not connected. Hence S_2 must consist of only one point. Therefore $S = P_1 + P_2 + P_3$.

Case II. Suppose P_1 and P_3 are in S_1 and P_2 is in S_2. Then

$$r_1 \text{ is } S_1 - P_1 \text{ or } S_2, \qquad r_2 \text{ is } S_2 - P_2 \text{ or } S_1, \qquad r_3 \text{ is } S_1 - P_3 \text{ or } S_2.$$

But r_1, r_2 and r_3 are mutually exclusive. Hence r_1 is $S_1 - P_1$, r_2 is $S_2 - P_2$ and r_3 is $S_1 - P_3$. Hence $S_1 - P_1$ and $S_1 - P_3$ have no point in common and therefore S_1 is $P_1 + P_3$ and is therefore not connected. Hence S_2 contains only one point, P_2. Therefore $S = P_1 + P_3 + P_2$. Our theorem is thus established.

It follows from the above theorem that if Axioms 5 and 6 are false and Axioms 1, 3 and 4 are true, then Axiom 7 is necessarily true. Hence there exist no examples of type E_{567} or E_{2567}.

That the set of Axioms 1–8 is independent in the sense that no one of these axioms follows from the others is shown by the examples E_1, E_2, E_3, E_4, E_5, E_6 and E_7 together with the following example, E_8.

E_8. S is a single point. This point is a limit point of no point-set.

§ 6. Concerning Axiom 4.

For Axiom 4 may be substituted the following two Axioms, 4(*a*) and 4(*b*).

AXIOM 4(*a*). *If P is a point and S is the set of all points, then $S - P$ is not connected.*

AXIOM 4(*b*). *If P is a point and S is the set of all points, then $S - P$ is composed of two connected subsets.*

THEOREM 17. *Axiom 4 is a consequence of Axioms 1, 4(a) and 4(b).*

Proof. By Axiom 4(*b*), $S - P$ is composed of two connected subsets, S_1 and S_2. By Axiom 4(*a*), $S - P$ is composed of two subsets M_1 and M_2 neither of which contains a limit point of the other. Suppose S_1 contains a point of M_1 and also a point of M_2. Let S_1' denote the set of all points of M_1 which lie in S_1 and let S_2' denote the set of all points of M_2 which lie in S_1. Then, since S_1 is connected, one of the sets S_1' and S_2' must contain a limit point of the other one and therefore, by Axiom 1, one of the sets M_1 and M_2 must contain a limit point of the other one. But this is contrary to hypothesis. It follows that one of the sets S_1 and S_2 coincides with M_1 and the other one coincides with M_2. Hence neither of the sets S_1 and S_2 contains a limit point of the other one. Thus Axiom 4 is established.

Let G' denote the set of Axioms 1–3, 5–8, 4(*a*) and 4(*b*).

To see that 4(*b*) is not a consequence of the remaining axioms of the set G', consider the example in which S denotes two intersecting straight lines in an ordinary euclidean plane and in which the terms point and limit have their usual significance. In this example, 4(b) is false but all the other axioms of the set G' are true.

The independence of (4a) in the set G' is exhibited by E_4.

It follows that G' is a categorical set of independent axioms.

§ 7. Concerning Categoricity.

In § 3 it was indicated that a set of axioms may be categorical with respect to certain defined terms but, at the same time, not *absolutely*[*] categorical. I will mention two interesting examples. Let H_{pl} denote the set of axioms composed of Hilbert's plane axioms of groups I and II together with a Dedikind cut continuity postulate and an axiom P to the effect that all points are coplanar. The set H_{pl} is not categorical with respect to the undefined symbols, point, line, association and betweenness, in terms of which these axioms are stated. Let V_{pl} denote the set of axioms composed of Veblen's Axioms I–VIII, XI together with axiom P (cf. above). The set V_{pl} is not categorical (with respect to point and order). But both H_{pl} and V_{pl} are categorical with respect to *point* and *limit* point of a point-set as defined in Veblen's Theory of Plane Curves in Non-Metrical Analysis Situs.[†]

The truth of the above statements concerning the sets H_{pl} and V_{pl} may be easily seen with the aid of results established in my paper, On a Set of Postulates Which Suffice to Define a Number-plane.[‡]

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[*] Cf. § 3.

[†] Transactions of the American Mathematical Society, Vol. 6 (1905), p. 85.

[‡] Ibid., Vol. XVI (1915), pp. 27–32.